* Rationalizing Substitutions:

Some nonrational functions can be changed into rational function by means of appropriate substitutions.

for examples

* $\int \frac{\sqrt{x+9}}{x}\,dx$, let $u=\sqrt{x+9} \Rightarrow u^2 = x+9$

$x = u^2-9$ and $dx = 2u\,du$

$$\int \frac{\sqrt{x+9}}{x}\,dx = \int \frac{u}{u^2-9}\,2u\,du = 2\int \frac{u^2}{u^2-9}\,dx$$

$$= 2\int \left(1+\frac{9}{u^2-9}\right)du = 2\int 1\,dx + 18\int \frac{1}{(u-3)(u+3)}\,dx$$

* $\int \frac{dx}{\sin x\,(1+\cos^2 x)} = I$ let $u=\cos x$, $du = -\sin x\,dx$

$dx = \frac{-1}{\sin x}\,du$

$$I = \int \frac{-1}{(1+u^2)(1-u^2)}\,du = \int \frac{-1}{(1-u)(1+u)(1+u^2)}\,du$$ $\sin^2 x = 1-\cos^2 x = 1-u^2$

* Evaluating integrals of the form

$\int \frac{1}{f(x)\sqrt{g(x)}}\,dx$, where $f(x)$ & $g(x)$ are linear or quadratic functions of x.

① $\int \frac{1}{\text{linear}\sqrt{\text{linear}}}\,dx$, put $\sqrt{\text{linear}} = z$

* $\int \frac{1}{(x+1)\sqrt{x+2}}\,dx$ let $z=\sqrt{x+2}$ $x = z^2-2$

$dx = 2z\,dz$

$$\int \frac{1}{(z^2-2+1)}\frac{2z}{z}\,dz = 2\int \frac{1}{z^2-1}\,dz = 2\int \frac{1}{(z+1)(z-1)}\,dz$$

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$$\frac{1}{z^2-1}=\frac{1}{(z+1)(z-1)}=\frac{A}{z+1}+\frac{B}{z-1}$$

(2) $\int \frac{1}{\text{Quadratic}\sqrt{\text{linear}}}\,dx$, put $z=\sqrt{\text{linear}}$

eg $\int \frac{x}{(x^2-3x+2)\sqrt{x-1}}\,dx$, Let $z=\sqrt{x-1}$

$(x-1)(x-2)$ $\quad x=z^2+1 \Rightarrow dx=2z\,dz$

$$\int \frac{z^2+1}{z^2(z^2-1)z}\cdot 2z\,dz$$

$$=2\int \frac{z^2+1}{z^2(z^2-1)}\,dz \quad , \quad \frac{z^2+1}{z^2(z+1)(z-1)}=\frac{A}{z}+\frac{B}{z^2}+\frac{C}{z+1}+\frac{D}{z-1}$$

* Note that when the integrand contains only one radical $(ax+b)^{\frac{1}{n}}$ $n>1$, put $z=\sqrt[n]{ax+b}$.
* if the linear $ax+b$ has two different power, then use the least common ~~divisor~~ multiple between them

means put $z=\sqrt[\text{big power}]{ax+b}$

eg $\int \frac{dx}{\sqrt{x+1}-\sqrt[4]{x+1}} = \int \frac{dx}{(x+1)^{\frac{1}{2}}-(x+1)^{\frac{1}{4}}}$

The L.C.M. of 2 & 4 is 4.

$z=\sqrt[4]{x+1} \Rightarrow z^4=x+1 \Rightarrow x=z^4-1 \quad , dx=4z^3\,dz$

$$\int \frac{4z^3\,dz}{z^2-z}=4\int \frac{\cancel{z}\cdot z^2}{\cancel{z}(z-1)}\,dz=4\int \frac{z^2-1+1}{z-1}\,dz=4\int (z+1)+\frac{1}{z-1}\,dz$$

$$=4\left[\frac{z^2}{2}+z+\ln|z-1|\right]+c$$