## 4.2 Definite Integral

When we compute the area under a curve, we obtained a limit of the form

$$\lim_{n\to\infty}\sum_{x=1}^{n} f(x_i^*)\Delta x.$$

This same limit shows up when we consider finding the distance given the velocity, and it turns our that this limit shows up in plenty of other situations. As it will appear for the next, ohhh, forever, we give it a special definition:

**Definition 4.2.** If $f$ is a function defined over the interval $[a,b]$, we divide $[a,b]$ into $n$ subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a, x_a, x_2, ..., x_{n-1}, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, ..., x_n^*$ be any sample points in the subintervals, so $x_i^* \in [x_{i-1}, x_i]$.

Then, the **definite integral** of $f$ from $a$ to $b$ is

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x,$$

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x \text{ for right-hand endpoints}$$

provided that this limit exists and gives the same value for all possible choices of sample points. If this limit does exist, we say that $f$ is **integrable** over $[a,b]$.

For notation,

$$\int_a^b f(x)dx,$$

we say that $\int$ is the integral sign, $a$ is the lower limit and $b$ is the upper limit of integration. The $dx$ indicates that the independent variable is $x$ – all other variables may be treated as constants. Also, the integral $\int_a^b f(x)dx$ is a number – it doesn not depend on $x$, and there is nothing special about $x$.

For the function $f$, we have

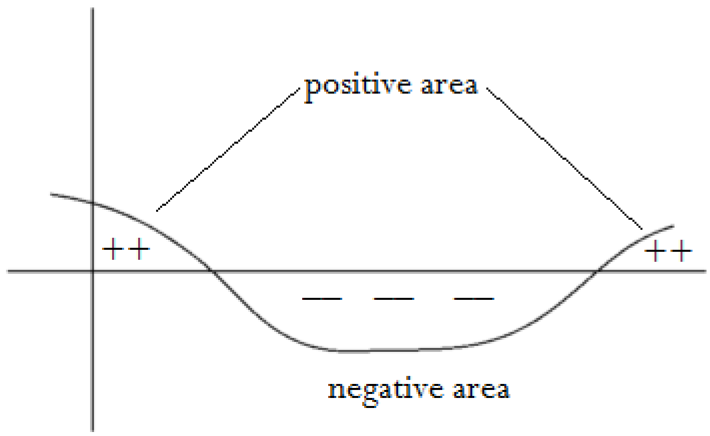
$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(y)dy,$$

for any variable as a placeholder. Lastly, we say that the sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a Riemann sum. If our function of interest, $f(x)$, is always non-negative, then we can treat the Riemann sum as a sum of areas of rectangles. Since the definite integral is the limit of a Riemann sum, a definite integral is the limit of the sum of area of rectangles, and thus the area under the curve $f(x)$ from $x = a$ to $x = b$.

If we consider a function $f$ that does take on negative values, such as a function like the one below,



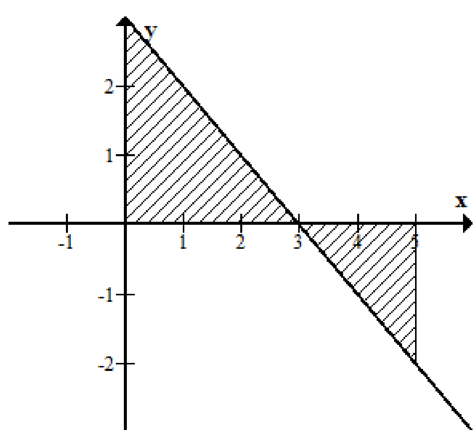

we cannot just take the heights of all the approximating rectangles – some of these heights would be negative, and that wouldn't make a whole lot of sense to have negative area. The way we can get around this is to consider the net area – we find the area of the curve above the $x$-axis. Then, separately, find the area of the curve under the $x$-axis. We then add these two areas together. So if $A_1$ is the area above the $x$-axis but under $f(x)$ computed by base times height, $\Delta x \cdot f(x_i^*)$ and $A_2$ is the area under the $x$-axis but above $f(x)$, also computed by base times height, $\Delta x \cdot f(x_i^*)$, we have

$$\text{Area of shaded region } = A_1 - A_2,$$

since $A_2$ is negative.

**Example 4.6.** Evaluate $\displaystyle\int_0^5 (3-x)\,dx$ and the area of the shaded region.

1. Let's take a look at the graph



Do you see how some of the shaded region is above the $x$-axis and some of it is below. When we evaluate $\int_0^5 (3-x)\,dx$, it would consider the area below negative. We can do it two different ways at this point. Let's do it without calculus.

2. Do you see how the two shaded regions are triangles? Guess what? We know the formula for the area of a triangle, $A = \frac{1}{2}bh$.

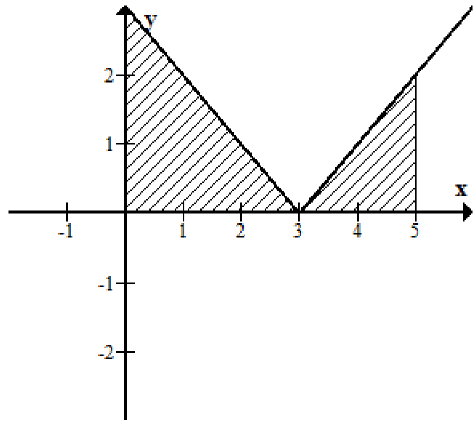
$$\text{Area of top triangle: } A = \frac{1}{2}(3)(3) = \frac{9}{2}$$
$$\text{Area of bottom triangle: } A = \frac{1}{2}(2)(2) = 2$$

So our integral is

$$\int_0^5 (3-x)\,dx = \frac{9}{2} - 2 = \frac{5}{2} = 2.5$$

3. Now if you want area of the shaded region (pretending like the area below the $x$-axis is also positive), then what we really want to evaluate is

$$\int_0^5 |3-x|\,dx$$




So the area of the shaded region is

$$\int_0^5 |3 - x|\, dx = \frac{9}{2} + 2 = \frac{13}{2} = 6.5$$

**Theorem 4.1.** *If $f$ is continuous on $[a, b]$, or if $f$ has only a finite number of jump discontinuities, then $f$ is integrable on $[a, b]$ – meaning that the definite integral $\int_a^b f(x)dx$ exists.*

This theorem is NOT easy to prove, by any stretch of the word easy. We do not do it here, but it does tell us something. It says that there are functions which are not integrable.

In order to simplify the calculations, we can choose specific sample points, and the right most endpoints are as good as any other. So, we have a half definition, half theorem:

**Defi-thereom:** If $f$ is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b - a}{n}$$

and $x_i = a + i\Delta x$.

In order to evaluate these integrals, we have to be able to work with some very commonly seen sums – the following three equations will be insanely valuable in doing this:

$$\begin{aligned}
\sum_{i=1}^{n} 1 =& n \\
\sum_{i=1}^{n} i =& \frac{n(n = 1)}{2} \\
\sum_{i=1}^{n} i^2 =& \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1}^{n} i^3 =& \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

The remaining rules that will help evaluate sums are very similar to rules that let us evaluate limits:

$$\begin{aligned}
\sum_{i=1}^{n} c =& nc \\
\sum_{i=1}^{n} ca_i =& c\sum_{i=1}^{n} a_i \\
\sum_{i=1}^{n}(a_i + b_i) =& \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \\
\sum_{i=1}^{n}(a_i - b_i) =& \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i
\end{aligned}$$

**Example 4.7.** Let's find the Riemann sum for $f(x) = 3 - x$ by taking right-hand endpoints over the interval [0,5]. By the way, we know the answer should be $\frac{5}{2}$. I'll take you through the procedure. Do this whenever you're asked to find evaluate the Riemann Sum.

1. Find $\Delta x$:

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}$$

2. Find $x_i$:

$$x_i = a + i\Delta x = 0 + i \cdot \frac{5}{n} = \frac{5i}{n}$$

3. Find $f(x_i)$:

$$\begin{aligned} f(x_i) &= 3 - x_i \\ &= 3 - \left(\frac{5i}{n}\right) \end{aligned}$$

4. Find $A_i$:

$$\begin{aligned} A_i &= f(x_i) \cdot \Delta x \\ &= \left(3 - \frac{5i}{n}\right) \cdot \frac{5}{n} \\ &= \frac{15}{n} - \frac{25i}{n^2} \end{aligned}$$

5. Find $\sum_{i=1}^{n} A_i$:

$$\begin{aligned} \sum_{i=1}^{n} A_i &= \sum_{i=1}^{n} \frac{15}{n} - \frac{25i}{n^2} \\ &= \frac{15}{n} \sum_{i=1}^{n} 1 - \frac{25}{n^2} \sum_{i=1}^{n} i \\ &= \frac{15}{n} \cdot n - \frac{25}{n^2} \cdot \left(\frac{n(n+1)}{2}\right) \\ &= 15 - \frac{25n(n+1)}{2n^2} \end{aligned}$$

6. Our final step is take the limit as $n \to \infty$.

$$\text{True Area } = \lim_{n\to\infty} 15 - \frac{25n(n+1)}{2n^2} = 15 - \frac{25}{2} = 2.5$$

which is exactly what we got from the before.

**Example 4.8.** Find the Riemann sum for $f(x) = 2x^3 - 4x$ by taking sample points to be right endpoints, where $a = 0$, $b = 2$ and $n = 4$.

With $n = 4$, the interval width is

$$\frac{2-0}{4} = \frac{1}{2},$$

and the right endpoints are $x_1 = .5$, $x_2 = 1$, $x_3 = 1.5$ and $x_4 = 2$. The Riemann sum is

$$\begin{aligned} R_4 =& \sum_{i=1}^{4} f(x_i)\Delta x \\ =& \Delta x \left(f(0.5) + f(1) + f(1.5) + f(2)\right) \\ =& \frac{1}{2}\left(\left(\frac{1}{4} - 2\right) + (2-4) + \left(\frac{27}{4} - 6\right) + (16-8)\right) \\ =& \frac{5}{2} \end{aligned}$$

Note that since this function does dip negative, this value is NOT the approximation for the area under the curve. However, it does represent the difference in positive and negative areas of the approximating rectangles of the curve. But, this is just an approximation. Now, let's evaluate

$$\int_0^2 2x^3 - 4x dx.$$

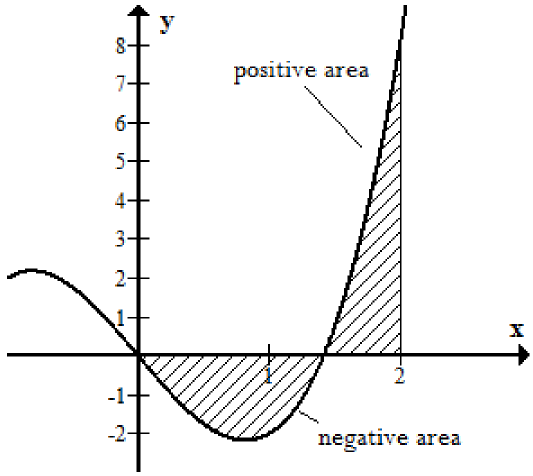
With $n$ subintervals, we have

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}.$$

and we have $x_0 = 0$, $x_1 = 2/n$, $x_2 = 4/n$, and in general, $x_i = 2i/n$. We use right endpoints and our theorem to let us evaluate

$$\begin{aligned}
\int_0^2 2x^3 - 4x dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right) \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \frac{2}{n} \cdot \sum_{i=1}^{n} \left(2\left(\frac{2i}{n}\right)^3 - 4\left(\frac{2i}{n}\right)\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \cdot \sum_{i=1}^{n} \left(\frac{16i^3}{n^3} - \frac{8i}{n}\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \cdot \left(\frac{16}{n^3} \sum_{i=1}^{n} i^3 - \frac{8}{n} \sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \cdot \left(\frac{16}{n^3} \cdot \left(\frac{n(n+1)}{2}\right)^2 - \frac{8}{n} \cdot \left(\frac{n(n+1)}{2}\right)\right) \\
&= \lim_{n\to\infty} \frac{2}{n} \cdot \left(\frac{4(n+1)^2}{n} - 4(n+1)\right) \\
&=8 \lim_{n\to\infty} \frac{(n+1)^2}{n^2} - \frac{n+1}{n} \\
&=8 \cdot \lim_{n\to\infty} \frac{1 - 2/n - 1/n^2}{1} - \frac{1+1/n}{1} \\
&=8 \cdot (1-1) \\
&=0
\end{aligned}$$

Clearly, this integral cannot be interpreted as the area under a curve, since this curve clearly does not have 0 area. However, it can be interpreted as a difference between the positive and negative area. If we graph the function, we should see that the positive and negative areas are identical:




This is NOT how we will be computing these soon. But just like as with derivatives, we have to go through it the long way first before we get to the shortcuts.

**Example 4.9.** Set up an expression for $\int_3^7 x^7\,dx$ as a limit of sums. Do not evaluate.

We let $f(x) = x^7$, $a = 3$ and $b = 7$. Thus,

$$\Delta x = \frac{7-3}{n} = \frac{4}{n}.$$

We have $x_1 = 3 + 1 \cdot \frac{4}{n}$, $x_2 = 3 + 2 \cdot \frac{4}{n}$, ..., and we have a generic term

$$x_i = 3 + \frac{4i}{n}.$$

By our theorem, we have

$$\begin{aligned}\int_3^7 x^7 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(3 + \frac{4i}{n}\right) \cdot \frac{4}{n} \\ &= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left(3 + \frac{4i}{n}\right)^7\end{aligned}$$

This is not very easy. Soon enough we'll get to the shortcuts that allow us to evaluate this quickly. By the way,

$$\int_3^7 x^7\,dx = 719780$$