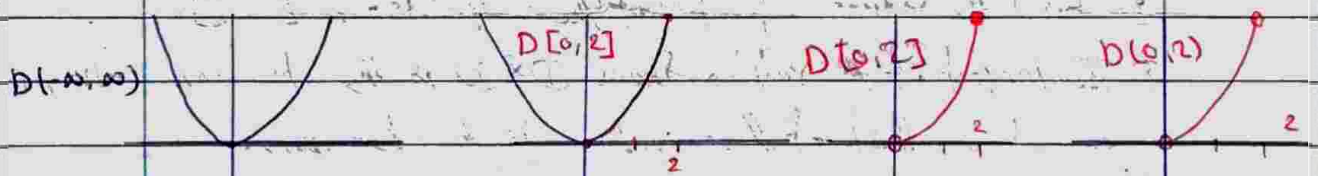
Applications of Derivatives

* Extreme Values of Functions

<u>Def:</u> Let f be a function with domain D. Then f has an absolute (or global) maximum at c if $f(c) \geq f(x), \forall x \in D$. The number $f(c)$ is called the maximum value of f on D. Similarly, f has an absolute minimum at c if $f(c) \leq f(x), \forall x \in D$, and $f(c)$ is called the minimum value of f on D.

* The maximum & minimum values of f are called the extreme values of f.

E.g. Consider the function $f(x) = x^2$ on different domains



only abs min "0". No absc max.	abs max = 4 at 2, abs min = 0 at 0	abs max = 4 at 2, No abs min.	no abs max & min.

<u>Thm:</u> (The Extreme Value Theorem).
If f is continuous on a closed interval $[a,b]$, then f has an absolute maximum value $f(c)$ & an absolute minimum value $f(d)$ at some $c, d \in [a,b]$.

<u>Def:</u> (Local Max/Min): A function f has a local (relative) max at c if $f(c) \geq f(x), \forall x$ in some open interval I containing c, and has a local min if $f(c) \leq f(x)\ \forall x \in I$.

e.g. ① $f(x)=x^2$, "0" $= f(0)$ is the absolute & local min. of f.
② $f(x)=\cos x$, $f(2n\pi)=1$ is the abs & local max. of f.

Fermat's Thm: If f has a local max or min at c, and if $f'(c)$ exists, then $f'(c)=0$.

Defi (Critical Point): A point c of the domain of a function f is called a critical point of f if $f'(c)=0$ either or undefined.

~~How to find abs max/min values of a continuous function~~ f on a closed interval $[a,b]$.
① Find the values of f at the critical points in (a,b).
② Find the values of f at the endpoints of $[a,b]$.
③ The largest of the values from ① & ② is the abs max value, and the smallest of these values is the abso min.

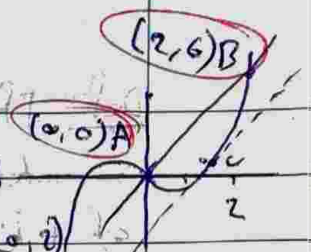
e.g. Find the extreme values of $f(x)=x^3-3x^2+1$ in $[-1,4]$
critical points are $x=0,2\in[-1,4]$ & then do the steps.

* Rolle's Theorem: Let f be a continuous function on the closed interval $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$, then $\exists$ $c\in(a,b)$ s.t. $f'(c)=0$.

The Mean Value Theorem: Let f be cont. on the $[a,b]$ & diff. on (a,b). Then $\exists$ $c\in(a,b)$ s.t.

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

To understand the role of this, we give an example



Eg. Consider $f(x) = x^3 - x$ on $[0, 2]$

Since $f(x)$ is poly, so it is cont on $[0, 2]$ & differentiable on $(0, 2)$. By M.V.T. $\exists\ c \in (0, 2)$

s.t. $f'(c) = \dfrac{f(2) - f(0)}{2 - 0}$.

Now

$f'(x) = 3x^2 - 1 \Rightarrow f'(c) = 3c^2 - 1$. & $f(2) = 6$ & $f(0) = 0$

$\Rightarrow \quad 3c^2 - 1 = \dfrac{6 - 0}{2 - 0} \rightarrow 6c^2 - 2 = 6 \Rightarrow c^2 = \dfrac{4}{3}$

$c = \mp \dfrac{2}{\sqrt{3}}$, $-\dfrac{2}{\sqrt{3}}$ will be ignored as it does not belong to $(0, 2)$. But $c = \dfrac{2}{\sqrt{3}} \in (0, 2)$. The tangent line at this c is parallel to the secant line AB (1)

* Thm (Increasing / Decreasing Test)

① If $f'(x) > 0$ on an interval I, then f is increasing on I.

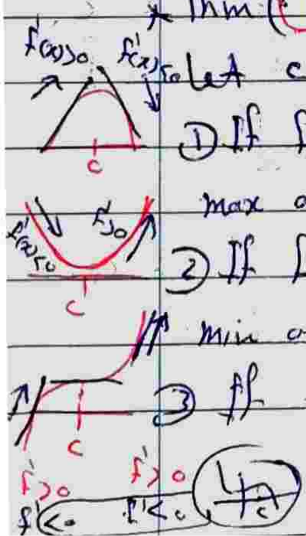
② If $f'(x) < 0$ on an interval I, then f is decreasing on I.

③ If $f'(x) = 0$ on an interval I, then f is constant on I.

Eg. Find the critical points & interval of increasing & decreasing of $f(x) = x^3 - 12x - 5$.

The First Derivative Test for

* Thm (Local Extrema).



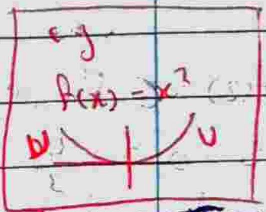
Let c be a critical point of a continuous function f.

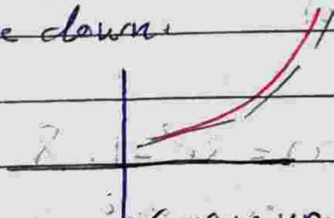
① If f' changes from +ve to −ve at c, then f has a local max at c.

② If f' changes from −ve to +ve at c, then f has a local min at c.

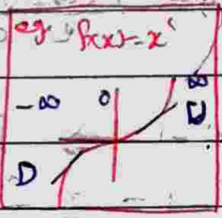
③ If f' does not change sign at c, then f has no local max or min.

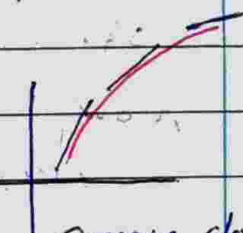
Def: If the graph of f lies above all of its tangents on an interval I, then it is called concave up. If the graph of f lies below all of its tangents on I, it is called concave down.





Concave up.





Concave down.

*Thm (Concavity Test). Let f be twice differentiable on an interval I.

1) If $f'' > 0$ on I, the graph of f is concave up on I.
2) If $f'' < 0$ on I, the graph of f is concave down on I.

Def: A point on the graph of f is called an inflection point if f is continuous there and the graph changes from concave up to concave down or the converse.

Thm: (The Second Derivative Test). Let f'' be continuous ~~in the neighborhood~~ on an open interval containing of points $(c, f(c))$.

1) If $f''(c) < 0$ & $f'(c) = 0$, then f has a local max at "c".
2) If $f''(c) > 0$ & $f'(c) = 0$, then f has a local min at "c".
3) If $f''(c) = 0$ & $f'(c) = 0$, the test fails. The function f may have a local max, a local min or neither.

Eg. Sketch the graph of $f(x) = x^4 - 4x^3$ using concavity, points of inflection and local maxima & minima.