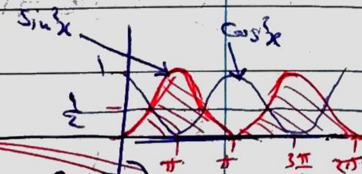
Applications of Integration

* Area between curves: In earlier chapter, we defined areas of regions that lie under the graphs of functions (between the graph of function and the x-axis). For example,

example:

Find the area under the curve $y = \sin^2 x$ on $[0, 2\pi]$.



$$A = \int_a^b f(x)\,dx \quad (\text{because } \sin^2 x \geq 0 \ \forall x \in [0, \pi])$$

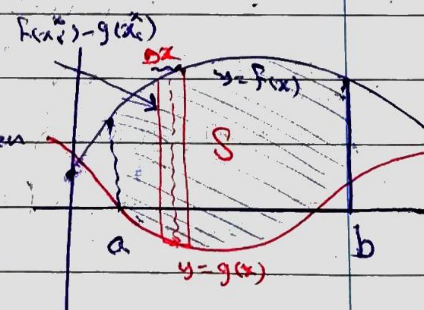
$$A = \int_0^{2\pi} \sin^2 x\,dx = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2x)\,dx$$

$$= \frac{x}{2} - \frac{1}{4}\sin 2x \Big|_0^{2\pi} = \left(\frac{2\pi}{2} - \frac{1}{4}\underbrace{\sin 4\pi}_{0}\right) - (0 - 0)$$

$$= \pi - 0 = \pi.$$

Now, we use integrals to find areas of regions that lie between the graphs of two functions

* Consider the region S that lie between $y = f(x)$ & $y = g(x)$ and between the vertical lines $x = a$ & $x = b$, where f & g are continuous and $f(x) \geq g(x)$, $\forall x \in [a, b]$.



As we did before (in definite integrals section), we divide S into n strips (parts) of equal width and then we approximate the i-th strip by a rectangle with base

Δx and height $f(x_i^*) - g(x_i^*)$. The Riemann Sum

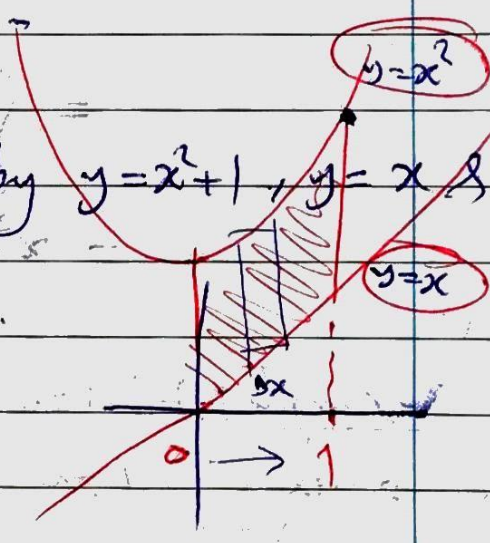
$\sum [f(x_i^*) - g(x_i^*)] \Delta x$ is approximately the area of S.
So the area A of S is

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x = \int_a^b [f(x) - g(x)] \, dx$$

Def: Let f & g be continuous with $f(x) \geq g(x)$ over $[a,b]$. The area of the region between curves $y=f(x)$ & $y=g(x)$ from a to b is

$$A = \int_a^b (f(x) - g(x)) \, dx.$$

Examples: Find the area bounded by $y = x^2+1$, $y = x$ & by lines $x=0$ & $x=1$.



$$A = \int_a^b (f(x) - g(x)) \, dx$$

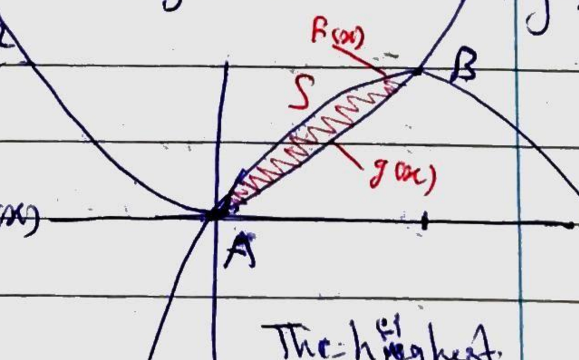
$$A = \int_0^1 [(x^2+1) - x] \, dx$$

$$= \frac{x^3}{3} + x - \frac{x^2}{2} \Big|_0^1 = \frac{1}{3} + 1 - \frac{1}{2} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

Example 2: Find the area of the region enclosed by the parabolas $y = x^2$ & $y = 2x - x^2$

$y = x^2 = f(x)$

$y = 2x - x^2 = -(x-1)^2 + 1 = f(x)$



The highest is f(x).

To find intersection points between f & g, equate both functions

$\Rightarrow$ $f(x) = g(x) \Rightarrow x^2 = 2x - x^2$

$2x^2 - 2x = 0$ ($\div 2$)

$x(x-1) = 0$

$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$

$x = 1 \Rightarrow y = 1 \Rightarrow (1,1)$ } are intersection points

So $a = 0$ & $b = 1$.

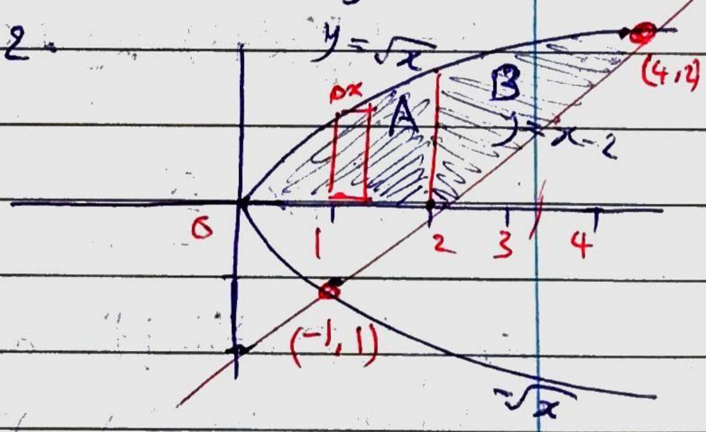
Now,

$$A = \int_a^b (f(x) - g(x))\,dx$$

$$= \int_0^1 [(2x - x^2) - x^2]\,dx = \int_0^1 (2x - 2x^2)\,dx$$

$$= x^2 - \frac{2}{3}x^3 \Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

example: Find the area bounded the curve $y = \sqrt{x}$, the x-axis and the line $y = x - 2$.



Find intersection points

$\sqrt{x} = x - 2$

$\Rightarrow x = (x-2)^2$

$\Rightarrow x = x^2 - 4x + 4$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$x = 1 \Rightarrow y = 1$ $(1,-1)$

$x = 4 \Rightarrow y = 2$ $(4,2)$ } are intersection points

When $x=1$, we will cancel it because it is below x-axis (Thus doesn't satisfy by $f(x)=\sqrt{x}$).

This ~~or~~ region will be divided into two subregions A & B.

from 0 to 2 $\Longrightarrow f(x)-g(x)=\sqrt{x}-0=\sqrt{x}$

& 2 to 4 $\Longrightarrow f(x)-g(x)=\sqrt{x}-(x-2)=\sqrt{x}-x+2$.

Total area = area A + area B

$$= \int_a^{\cancel{b}\,c} (f(x)-g(x))\,dx + \int_c^d (f(x)-g(x))\,dx$$

$$= \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x}-x+2)\,dx$$

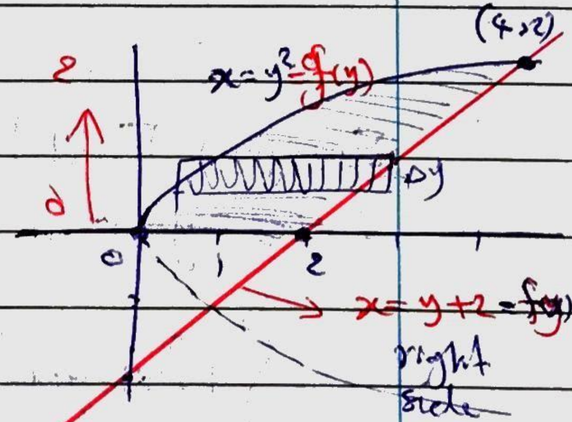
$$= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 + \frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2} + 2x\Big|_2^4$$

$$= \frac{10}{3}$$

Note that we can find this integration with respect to y.

$$A = \int_c^d [f(y)-g(y)]\,dy$$

function on the right   on the left



split y into strips.

$y=\sqrt{x} \Longrightarrow x=y^2 = g(y)$ left

$y=x-2 \Longrightarrow x=y+2 = f(y)$ right.

Find intersection points,

$y^2 = y+2 \Longrightarrow y^2-y-2=0$

$(y-2)(y+1)=0$

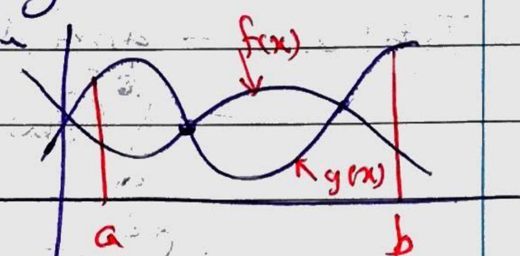
$y=-1$, $y=2$ $\Longrightarrow a=0,\ b=2$.

is below x-axis.

$$A=\int_0^2[(y+2)-y^2]\,dy=\int_0^2(y+2-y^2)\,dy$$

$$=\frac{y^2}{2}+2y-\frac{1}{3}y^3\Big|_0^2=\frac{4}{2}+4-\frac{8}{3}=\frac{18}{3}-\frac{8}{3}=\frac{10}{3}$$

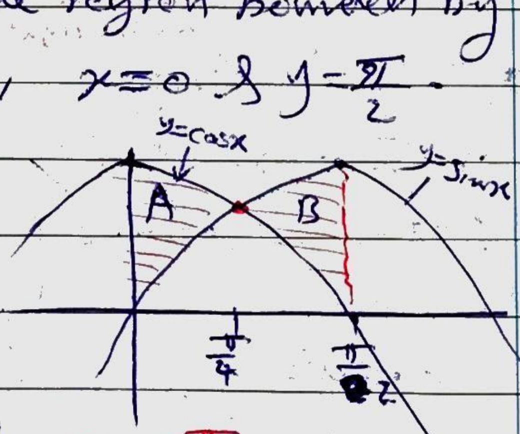
* If we have two functions f & g on [a,b] as in the following figure. These function change from heigh to low in [a,b], so the area between curves $y=f(x)$ & $y=g(x)$ from $x=a$ to $x=b$ is



$$A=\int_a^b|f(x)-g(x)|\,dx \quad \text{, where}$$

$$|f(x)-g(x)|=\begin{cases} f(x)-g(x) & \text{when } f(x)\geq g(x) \\ g(x)-f(x) & \text{when } g(x)\geq f(x). \end{cases}$$

<u>Example</u>: Find the area of the region bounded by the curves $y=\sin x$ & $y=\cos x$, $x=0$ & $y=\frac{\pi}{2}$.



$$A=\int_a^b|f(x)-g(x)|\,dx$$

$$=\int_0^{\frac{\pi}{2}}|\cos x-\sin x|\,dx$$

$$=\text{area } A+\text{area } B$$

$0\leq x\leq\frac{\pi}{4}$ $\cos x-\sin x$

$\frac{\pi}{4}\leq x\leq\frac{\pi}{2}$ $\sin x-\cos x$

$\boxed{\sin x\geq\cos x}$

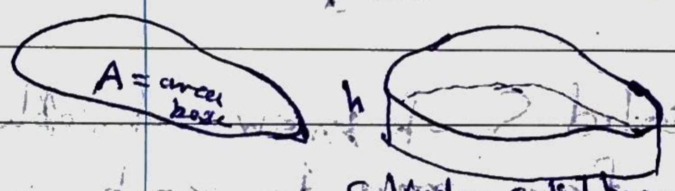
$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx$$

$$= \sin x + \cos x \Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

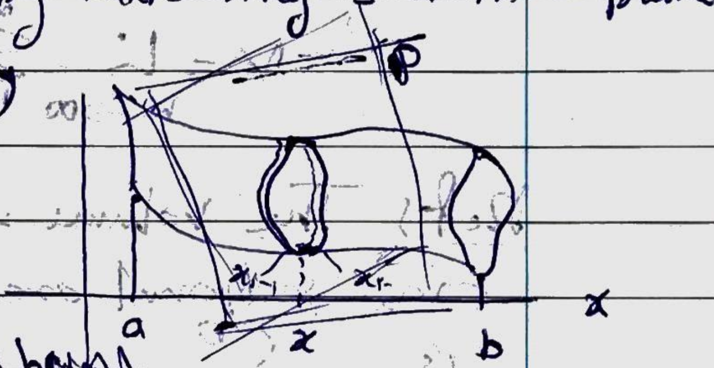
$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} - 2 = 2\sqrt{2} - 2.$$

Volumes:

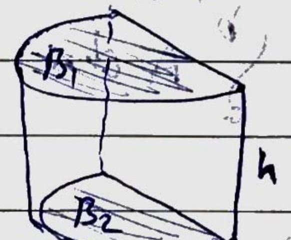
In this section, we define volumes of solids whose cross sections are plane regions. A cross section of a solid S is the plane region formed by intersecting S with a plane



Cylinder solid based on region Volume = base area x height



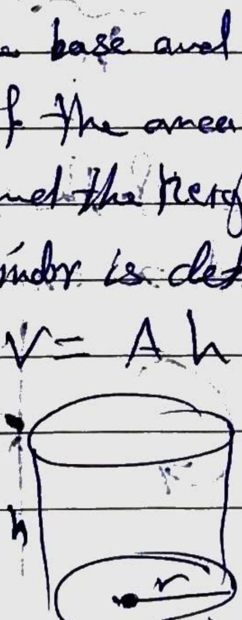
Suppose we want to find the volume of a solid S (like in the above figure). We start with a simple type of solid called a cylinder. The cylinder consists of all points on the line segment that are perpendicular to the base and join B_1 to B_2. If the area of the base is A and the height of the cylinder is h, then the volume V of the cylinder is defined as

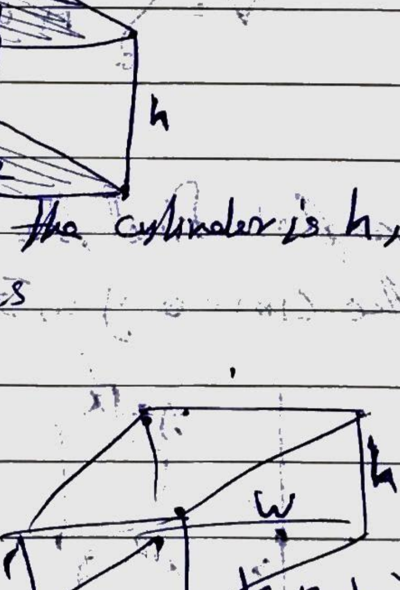


$$V = Ah.$$

In particular:

Circular cylinder $V = \pi r^2 h$.





Rectangular box $V = L w h$.

For a solid S that is not a cylinder, we cut S into pieces and approximate each piece by a cylinder. We estimate the volume of S by adding the volumes of the cylinders. Then we obtain the exact volume by limiting process in which the numbers of pieces becomes large.

Volume of i-th part $V_i = A(x_i)\,\Delta x_i$

The volume of the entire solid S is

$$V \simeq \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} A(x_i)\,\Delta x_i$$

This is the Riemann Sum of $A(x)$ on $[a,b]$.

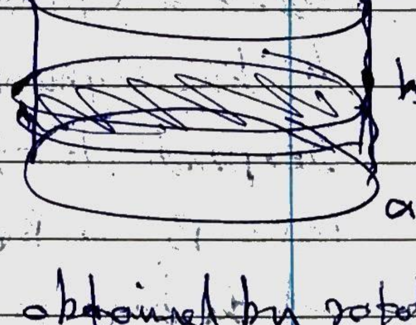
$$\Rightarrow V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\,\Delta x_i = \int_a^b A(x)\,dx.$$

Def: The volume of a solid S of known integrable cross sectional area $A(x)$ from $x=a$ to $x=b$ is

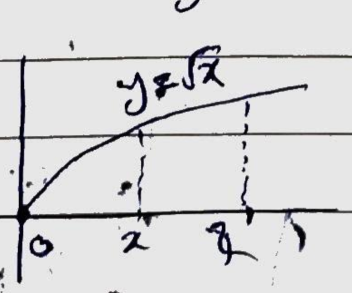
$$V = \int_a^b A(x)\,dx.$$

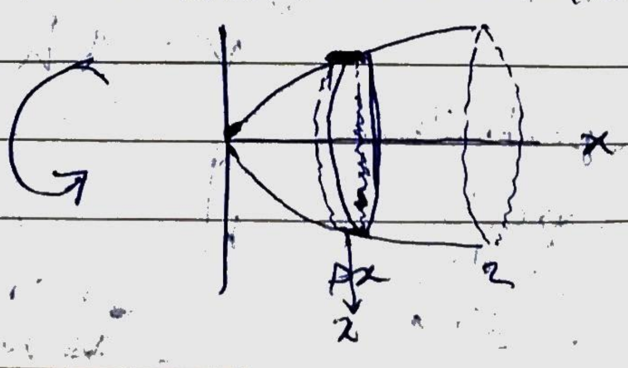
For a cylinder, the cross sectional is constant A, so

$$V = \int_a^b A\,dx = A(\underbrace{b-a}_{h}) = Ah$$



example: find the volume of the solid obtained by rotating the curve $y=\sqrt{x}$ about x-axis from 0 to 2.





If we slice the solid at x, we get a disk with radius $\sqrt{x}$. The area of cross section is

$$A(x)=\pi(\sqrt{x})^2=\pi x.$$

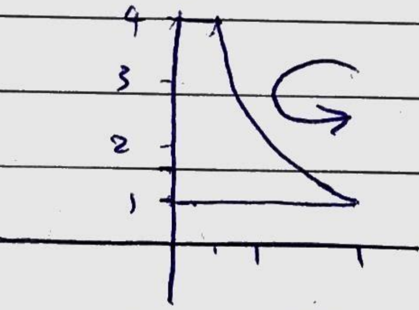
$$V=\int_a^b A(x)\,dx=\int_0^2 \pi x\,dx=\pi\frac{x^2}{2}\Big|_0^2=2\pi.$$

H.W.: Rotate about the line $y=1$. & V?

2) Find the volume of the solid generated by rotating the region bounded by the curve $x=\frac{2}{y}$ and $1\le y\le 4$, about y-axis.

$y=1 \Rightarrow x=\frac{2}{1}=2$

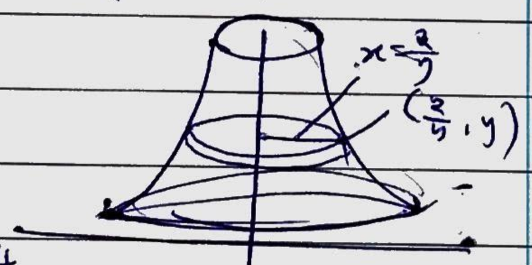
$y=4 \Rightarrow x=\frac{2}{4}=\frac{1}{2}$



$$V=\int_a^b A(x)\,dx$$

$$=\int_1^4 \pi\left(\frac{2}{y}\right)^2 dy$$

$$=4\int_1^4 \frac{\pi}{y^2}\,dy=4\pi\left(\frac{-1}{y}\right)\Big|_1^4=4\pi\frac{3}{4}=3\pi$$



3) Find the volume of a sphere of radius r.

$y=\sqrt{r^2-x^2}$, by Pythagorean Thm

or each slice is a circle of radius

$$x^2+y^2=r^2$$

$$y^2=r^2-x^2$$

$$V=\int_a^b A(x)\,dx=\int_{-r}^{r}\pi y^2$$

$$=\int_{-r}^{r}\pi(r^2-x^2)\,dx=2\pi\int_0^r (r^2-x^2)\,dx=2\pi\left(r^2x-\frac{x^3}{3}\right)\Big|_0^r=2\pi\left(r^3-\frac{r^3}{3}\right)=\frac{4}{3}\pi r^3.$$