* Integration of Rational Functions

we know that the rational function has the form

$$f(x) = \frac{P(x)}{Q(x)}, \quad \text{where } P \,\&\, Q \text{ are polynomials}$$

we consider two cases

① if $\deg(P) \geq \deg(Q)$, then we use the Long division Method to transfer $f(x)$ into sum of simpler functions

For example: Find

$$\int \frac{x^3+x}{x-1}\,dx = \int \left(\text{result} + \frac{\text{remainder}}{\text{divisor}}\right) dx$$

$$\Rightarrow \int \frac{x^3+x}{x-1}\,dx = \int \left(x^2+x+2+\frac{2}{x-1}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

divisor $\boxed{x-1}$, result $\boxed{x^2+x+2}$

$$\begin{array}{r} x^3 + x \\ \ominus\, x^3 \oplus x^2 \\ \hline x^2 + x \\ \ominus\, x^2 \oplus x \\ \hline 2x \\ \ominus\, 2x \oplus 2 \\ \hline \boxed{2} \end{array}$$

remainder

② if $\deg(P) < \deg(Q)$, use partial fractional Method.

* Partial Fractional Method:

Notice that, if

Left $\boxed{\dfrac{2}{x+1} + \dfrac{3}{x-3}} = \dfrac{2(x-3)+3(x+1)}{(x+1)(x-3)} = \boxed{\dfrac{5x-3}{x^2-2x-3}}$ Right

From this equation, we have

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3}\right) dx$$

$$= \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + c$$

We can easily move from Left to Right. How about the reverse. The partial fractional Method helps us to move from Right to Left.

* Suppose $\dfrac{5x-3}{x^2-2x-3} = \dfrac{5x-3}{(x+1)(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{x-3}$ ——— (1)

The above eq. with finding the values of A & B ~~[illegible]~~ is PFM.

We now find, A & B. Simplifying eq (1), we get

$$5x-3 = A(x-3) + B(x+1). \text{ Now}$$

If $x=-1 \Rightarrow 5(-1)-3 = A(-1-3) \Rightarrow -8 = -4A \Rightarrow A = 2.$

If $x=3 \Rightarrow 5(3)-3 = B(3+1) \Rightarrow 12 = 4B \Rightarrow B = 3.$

Another way of find A & B:

$$5x-3 = A(x-3) + B(x+1)$$

$$5x-3 = Ax - 3A + Bx + B$$

$\Rightarrow$

$$\Rightarrow 5x-3 = (A+B)x + (-3A+B)$$

$$\left.\begin{array}{ll} A+B = 5 & \longrightarrow (1) \\ -3A+B = -3 & \longrightarrow (2) \end{array}\right] \text{Solve.}$$

* We consider different cases of $Q(x)$.

(i) If $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$, then

$\exists\ A_1, A_2, \ldots, A_k$ s.t.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

e.g. $\displaystyle\int \frac{3x+11}{x^2-x-6}\,dx$

$$\frac{3x+11}{x^2-x-6} = \frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$\Rightarrow \quad 3x+11 = A(x+2) \pm B(x-3)$$

if $x=-2 \Rightarrow 3(-2)+11 = B(-2-3)$

$\Rightarrow 5 = -5B \Rightarrow B = -1.$

if $x=3 \Rightarrow 3(3)+11=A(3+2) \Rightarrow 20=5A \Rightarrow A=4.$

$$\int \frac{3x+11}{x^2-x+} dx = \int \frac{4}{x-3} dx - \int \frac{1}{x+2} dx$$

$$= 4\ln|x-3| - \ln|x+2| + C.$$

H.w.

(1) $\int \frac{x^2}{x^2-1} dx$ (2) $\int \frac{x-1}{x^2+x} dx$

(3) $\int \frac{6}{x^2-1} dx$

e.g.

Solve $\int \frac{\sqrt{x+1}}{x} dx$

Let $u=\sqrt{x+1} \Rightarrow u^2=x+1 \Rightarrow x=u^2-1$

$dx = 2u\,du.$

So $\int \frac{\sqrt{x+1}}{x} dx = \int \frac{u}{u^2-1} 2u\,du = 2\int \frac{u^2}{u^2-1} du$

Solve it as H.w (1)

$$\int \frac{u^2}{u^2-1} du = \int 1\, du - \int \frac{1}{u^2-1} du$$

(ii) if $Q(x) = \underbrace{(a_1x+b_1)^n}_{\text{repeated } n\text{-times}}(a_2x+b_2)\cdots(a_kx+b_k)$, then

$$\frac{P(x)}{Q(x)} = \underline{\frac{A_{11}}{(a_1x+b_1)} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1n}}{(a_1x+b_1)^n}} + \frac{A_2}{(a_2x+b_2)} + \cdots + \frac{A_k}{a_kx+b_k}$$

e.g. $\displaystyle\int \frac{4x}{x^3-x^2-x+1}\,dx$

$$x^3-x^2-x+1 = x^2(x-1)-(x-1) = (x-1)(x^2-1)$$
$$= (x-1)(x+1)(x-1) = (x-1)^2(x+1)$$

$$\Rightarrow \frac{4x}{x^3-x^2-x+1} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

If $x=1 \Rightarrow 4(1) = B(1+1) \Rightarrow B=2$

If $x=-1 \Rightarrow -4 = C(-2)^2 \Rightarrow C=-1$

If $x=0 \Rightarrow 0 = -A+B+C$

$$\Rightarrow A = B+C = 2+(-1) = 1$$

$$\int \frac{4x}{x^3-x^2-x+1}\,dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1}\right)dx$$
$$= \int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx - \int \frac{1}{x+1}\,dx$$
$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C.$$

(iii) if $Q(x) = (a_1x^2+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$,

then

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{\underbrace{(a_1x^2+b_1)}_{\text{irreducible}}} + \frac{A_2}{(a_2x+b_2)} + \cdots + \frac{A_k}{(a_kx+b_k)}$$

⊛ irreducible mean $a_1x^2+b_1 \neq 0 \ \forall x \in \mathbb{R}$.

e.g. $\displaystyle\int \frac{x-1}{(x+1)(x^2+1)}\,dx$

$$\frac{x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$\Rightarrow \quad x-1 = (x^2+1)A + (x+1)(Bx+C)$

If $x=-1 \Rightarrow -2 = 2A \Rightarrow A = -1$.

If $x=0 \Rightarrow -1 = A+C \Rightarrow -1 = -1+C \Rightarrow C=0$.

If $x=1 \Rightarrow 0 = 2A+2B \Rightarrow B = -A = 1$.

$$\Rightarrow \int \frac{x-1}{(x+1)(x^2+1)}\,dx = \int \frac{-1}{x+1} + \int \frac{x}{x^2+1}\,dx$$

$$= -\ln|x+1| + \frac{1}{2}\ln(x^2+1) + C$$

(iv) if $Q(x) = (a_1x^2+b_1)^n (a_2x+b_2) \cdots (a_kx+b_k)$,
then

$$\frac{P(x)}{Q(x)} = \frac{A_{11}x+B_1}{a_1x^2+b_1} + \frac{A_{12}x+B_2}{(a_1x^2+b_1)^2} + \cdots + \frac{A_{1n}x+B_n}{(a_1x+b_1)^n} + \frac{A_2}{(a_2x+b_2)}$$

$$+ \cdots + \frac{A_k}{(a_kx+b_k}.$$

e.g. $\displaystyle\int \frac{x+2}{(x+1)(x^2+1)^2}\,dx$

$$\frac{x+2}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$x+2 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1) + (Dx+E)(x+1)$$

The rest is H.w.

Answer: $A = \frac{1}{4}$, $B = \frac{-1}{4}$, $C = \frac{1}{4}$

$D = \frac{-1}{2}$, $E = \frac{3}{2}$.