

* Trigonometric Substitutions

Trig substitutions can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

Expression	substitution	identity
① $\sqrt{a^2 - x^2}$	$x = a \sin \theta, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$1 = \sin^2 \theta + \cos^2 \theta$
② $\sqrt{a^2 + x^2}$	$x = a \tan \theta, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ or $x = a \sinh \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
③ $\sqrt{x^2 - a^2}$	$x = a \sec \theta, \theta \in [0, \frac{\pi}{2})$ or $x = a \cosh \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$

eg $\int \frac{dx}{\sqrt{4+x^2}}$, let $x = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4+4 \tan^2 \theta}} = \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4(1+\tan^2 \theta)}} = \int \frac{2 \sec^2 \theta d\theta}{2 \sqrt{\sec^2 \theta}}$$

$$= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$$

$$\begin{aligned} x = 2 \tan \theta &\Rightarrow \tan \theta = \frac{x}{2} \\ 4 + x^2 &= 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta \\ \sec^2 \theta &= \frac{4+x^2}{4} \Rightarrow \sec \theta = \frac{\sqrt{4+x^2}}{2} \quad \sec \theta > 0 \quad \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ &\Rightarrow \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + C \end{aligned}$$

* $\int \frac{x^2}{\sqrt{9-x^2}} dx$, let $x = 3 \sin \theta$, $dx = 3 \cos \theta d\theta$,
 $9 - x^2 = 9 - 9 \sin^2 \theta = 9(1 - \sin^2 \theta) = 9 \cos^2 \theta$
 $= \int \frac{9 \sin^2 \theta \cdot 3 \cos \theta d\theta}{3 \cos \theta} = 9 \int \sin^2 \theta d\theta = 9 \int \frac{1 - \cos 2\theta}{2} d\theta$

$$= \frac{9}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) + C = \frac{9}{2} (\theta - \sin \theta \cos \theta) + C$$

$$= \frac{9}{2} \left(\sin^{-1} \frac{x}{3} - \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3} \right) + C$$

$$\theta = \sin^{-1} \frac{x}{3}, \quad \sin \theta = \frac{x}{3}, \quad \cos \theta = \frac{\sqrt{9-x^2}}{3}$$