Techniques of Integration.

① u Substitution Method.

Let $u = g(x)$ be a differential function whose range is an interval I and f is continuous on I. Then

$$\int f(g(x)) \cdot \underline{g'(x)\,dx} = \int f(u)\,\underline{du}.$$

$$\left(\frac{du}{dx} = g'(x) \Rightarrow du = \underline{g'(x)\,dx} \right).$$

e.g. Solve

* $\int \cos(5x-2)\,dx$, let $u = 5x-2$

$$du = 5\,dx \Rightarrow dx = \frac{1}{5}du$$

$$= \int \cos u \cdot \frac{1}{5}\,du = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + c$$

$$= \frac{1}{5}\sin(5x-2) + C.$$

* $\int \frac{\sin^2(\frac{1}{x})}{x^2}\,dx$, let $u = \frac{1}{x} \Rightarrow \frac{du}{dx} = \frac{-1}{x^2} \Rightarrow dx = -x^2\,du$

$$= -\int \sin^2 u\,du = \int \frac{\cos 2u - 1}{2}\,du = \frac{1}{2}\int(\cos 2u - 1)\,du$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin 2u - u\right) + c = \frac{1}{4}\sin 2u - \frac{1}{2}u + C$$

$$= \frac{1}{4}\sin\left(\frac{2}{x}\right) - \frac{1}{2x} + C.$$

* $\int \tan x\,dx = -\ln\cos x + C = \ln\sec x + C.$

* $\int_0^1 3x^2\sqrt{x^3+1}\,dx$ Solve with less steps

$= \frac{2}{3}(x^3+1)^{\frac{3}{2}}\Big|_0^1 = \frac{4}{3}\sqrt{2}-\frac{2}{3}$.

* $\int \frac{e^{2\tan^{-1}x}}{1+x^2}\,dx$, let $u=\tan^{-1}x$,

* $\int \sec x\,dx = \int \sec x \cdot \frac{\sec x+\tan x}{\sec x+\tan x}\,dx$ let $u=\sec x+\tan x$.

$du = \sec^2 x+\tan x \sec x\,dx$

$= \int \frac{\sec^2 x+\tan x\sec x}{\sec x+\tan x}\,dx = \ln|\sec x+\tan x|+C$.

eg $\int \frac{1}{x\ln x}\,dx$, $u=\ln x$.

② Integration by parts:

let u & v be differentiable of x, the product rule say

$$\frac{d}{dx}(u(x)\cdot v(x)) = u(x)\cdot v'(x) + v(x)\cdot u'(x)$$

$$\int \cancel{\frac{d}{dx}}(u\cdot v)\,\cancel{dx} = \int (u\cdot v' + v\cdot u')\,dx$$

$$u\cdot v = \int u\underline{v'}\,dx + \int v\cdot u'\,dx.$$

$v' = \frac{dv}{dx}\cdot dx$

$u' = \frac{du}{dx}\cdot dx$

$$u\cdot v = \int u\,dv + \int v\,du$$

$$\boxed{\int u\,dv = u\cdot v - \int v\,du}$$

* $\int \ln x\,dx$, $u=\ln x$ $dv=dx$, $v=x$

$du = \frac{1}{x}\,dx$

$$\int \ln x\,dx = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - x + C.$$

* $\int x\cos x\,dx = x\sin x + \cos x + C$

let $u = x \Rightarrow du = dx$

$dv = \cos x\,dx \Rightarrow v = \sin x$

* $\int x e^{x} dx$, let $u = x$, $du = dx$

$dv = e^{x}$, $v = e^{x}$.

$\int x e^{x} dx = x e^{x} - \int e^{x} dx = x e^{x} - e^{x} = e^{x}(x-1)$.

* $\int \sin^{-1}x\,dx$ let $u = \sin^{-1}x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}\,dx$

$dv = dx \Rightarrow v = x$.

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}$.

H.W $\int \tan^{-1}x\,dx$.

$\int e^{x}\sin x\,dx$, $\int x^2 e^{x} dx$, $\int \frac{\ln x}{x^2}\,dx$.

$\int x\tan^{-1}x\,dx$.

For integral Definite the rule is like this

* $\int_a^b u\,dv = u.v \Big|_a^b - \int_a^b v\,du$

$u(x)\ 'v(x)$ $\quad u(x).v(x)$ $\quad v(x) \cdot u'(x)$.

eg.. $\int_0^1 \tan^{-1}x\,dx$ let $u = \tan^{-1}x \Rightarrow du = \frac{1}{1+x^2}\,dx$

$dv = dx \Rightarrow v = x$.

$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$

$$=[1\cdot \tan^{-1}1 - 0\cdot\tan^{-1}0] - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx \qquad \ln|1+x^2|$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx \qquad \frac{1}{2}\ln 2 \qquad = \frac{\pi}{4} - \frac{\ln 2}{2}$$

*

$$\int \sin^n x\,dx \quad , n \geq 2 , n \in \mathbb{Z}$$

$$\int \sin^{n-1}x \sin x\,dx \quad , \text{let } u = \sin^{n-1}x \longrightarrow du = (n-1)\sin^{(n-2)}x \cos x\,dx$$

$$dv = \sin x \Longrightarrow v = -\cos x .$$

$$\int \sin^n x\,dx = -\cos x \sin^{(n-1)}x + \int (n-1)\sin^{(n-2)}x \cos^2 x\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - \int (n-1)\sin^n x\,dx$$

move

$$\boxed{-n\int \sin^n x\,dx} + \int \sin^n x\,dx$$

$$1 = \frac{n}{n}\int \sin^n x\,dx = \frac{-\cos x \sin^{n-1}x}{n} + \frac{(n-1)}{n}\int \sin^{n-2}x\,dx$$

Compute $\int \sin^3 x\,dx = ?$ $\qquad \int e^{nx}\sin x\,dx$

H.W. $\int \cos^n x\,dx$, $\int \tan^n x\,dx$, $\int \sec^n x\,dx$, $\int \sin^{-n} x\,dx$

you need the following inorder to solve the above integrals

Title

(3) Trigonometric Integrations

Recall that $\sin x^2 + \cos x^2 = 1 \longrightarrow \tan^2 x = \sec^2 x - 1$

$\cos 2x = \cos^2 x - \sin^2 x \qquad \cot^2 x = \csc^2 x - 1$

$\sin 2x = 2\sin x \cos x$

$\sin^2 x = \frac{1}{2}(1-\cos 2x) \longrightarrow (1-\cos x) = 2\sin^2\frac{x}{2}$,

$\cos^2 x = \frac{1}{2}(1+\cos 2x) \qquad 1+\cos x = 2\cos^2\frac{x}{2}$

$$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C .$$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$$

$$\int \sin(2\sin^{-1}x)\,dx$$

Trigonometric Integrals

* $\int \cot^2 x\,dx = \int (\csc^2 x - 1)\,dx = -\cot x - x + C.$

* $\int \frac{1}{1+\sin x}\,dx = \int \left(\frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}\right)dx = \int \frac{1-\sin x}{1-\sin^2 x}\,dx$

$$= \int \frac{1-\sin x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos x}\,\frac{1}{\cos x}\right)dx$$

$$= \int (\sec^2 x - \sec x \tan x)\,dx = \tan x - \sec x + C.$$

* $\int \frac{1}{1+\cos x}\,dx$ = by the way above $= \boxed{-\cot x + \csc x + C}$

OR ~ * prove that

$\int \frac{1}{1+\cos x}\,dx = \int \frac{1}{2\cos^2 \frac{x}{2}}\,dx = \int \frac{1}{2}\sec^2\frac{x}{2}\,dx = \boxed{\tan\frac{x}{2} + C}$

* $\int \sqrt{1+\cos x}\,dx = \int \sqrt{2\cos^2\frac{x}{2}}\,dx = \int \sqrt{2}\,.\cos\frac{x}{2}\,dx = 2\sqrt{2}\,\sin\frac{x}{2} + C.$

$\int \sqrt{1+\cos ax}\,dx$?

* $\int \sin^4 x\,dx$, we can use $\int \sin^n x\,dx$ as $n=4$ to compute this integeral, but we can compute it in different way?

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx = \frac{1}{4}\int \left[1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right]dx$$

$$= \frac{1}{4}\left(x - \sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)\right) = \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C.$$

* product of power of sine & cosine

How compute $\boxed{\int \sin^m x \cos^n x\,dx}$, $n, m \geq 0$, $m, n \in \mathbb{Z}$.

$\int$

(i) If the power of cosine is odd (i.e. $n=2k+1$), save one cosine factor and use $\cos^2 x = 1-\sin^2 x$ to express the remaining factors in terms of sine.

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x \underbrace{\cos^{2k} x}_{(\cos^2 x)^k} \cdot \cos x\, dx$$

$$= \int \sin^m x (1-\sin^2 x)^k \cos x\, dx.$$ Then substitute $u=\sin x$ if needed!!

(ii) If the power of sine is odd ($m=2k+1$), save one sine and use $\sin^2 x = 1-\cos^2 x$ to express the remaining factors in terms of cosine.

$$\int \sin^{2k+1} x \cos^n x\, dx = \int (\sin^2 x)^k \sin x \cos^n x\, dx$$

$$= \int (1-\cos^2 x)^k \cos^n x \sin x\, dx$$

Then put $u=\cos x$.

Note that if both m & n are odd, either (i) or (ii) can be used.

(iii) If ~~the~~ both m & n are even, then use

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \text{ and } \cos^2 x = \frac{1}{2}(1+\cos 2x).$$

- Sometimes you need to use

$$\sin x \cos x = \frac{1}{2}\sin 2x.$$

e.g. $$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos x^2 \sin x\, dx$$

$$= \int (1-\cos^2 x)\cos x^2 \sin x\, dx = \int \cos^2 x \sin x\, dx - \int \cos^4 x \sin x\, dx$$

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.$$

* $\int \sin^4 x \cos^3 x\,dx = \int \sin^4 x \cos^2 x \cos x\,dx$

$= \int \sin^4 x (1-\sin^2 x) \cos x\,dx$

$= \int \sin^4 x \cos x\,dx - \int \sin^6 x \cos x\,dx$

$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$

* $\int \cos^5 x\,dx$, $m=0,\ n=5.$

$\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx = \int (\cos^2 x)^2 \cos x\,dx = \int (1-\sin^2 x)^2 \cos x\,dx$

$= \int (1 - 2\sin^2 x + \sin^4 x)\cos x\,dx = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$

product of power of tangent & secant

* A way to find $\boxed{\int \tan^m x \sec^n x\,dx}$, $m, n \in \mathbb{Z}^+$.

① if n is even, $n = 2k$ $(k \geq 1)$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express factors in terms of $\tan x$

$\int \tan^m x \sec^{2k} x\,dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x\,dx$

put $u = \tan x$ $= \int \tan^m x (1+\tan^2 x)^{k-1} \sec^2 x\,dx.$

② if m is odd, $m = 2k+1$, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express factors in terms of $\sec x$.

$\int \tan^{2k+1} \sec^n x\,dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x\,dx.$

e.g. $\int \tan^5 x \sec^7 x\,dx = \int \tan^4 x \sec^6 x \sec x \tan x\,dx$ $= \int (\sec^2 x - 1)^k \sec^{n-1} x\, \sec x \tan x\,dx.$

By the same strategy one can find put $u = \sec x$

$\int \cot^m x \csc^n x\,dx.$ e.g $\int \tan^4 x \sec^4 x\,dx$

$= \int (\sec^2 x - 1)^2 \underbrace{\sec^6 x}_{u} \cdot \underbrace{\sec x \tan x\,dx}_{du} = \int (u^2-1)^2 u^6\,du$?

not even.   the so called

* $\int \sec^{3} x\,dx$. This integral can be solved by uv

~~by the formula given earlier~~

$u = \sec x \quad du = \sec x \tan \quad , \quad dv = \sec^2 x\,dx \ , \ v = \tan x$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 dx \qquad \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,d + \int \sec x\,dx$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C.$$

* $\int \tan^4 x\,dx = \int \tan^2 x \tan^2 x\,dx = \int \tan^2 x(\sec^2 x - 1)\,dx$

$$= \int (\tan^2 x \sec^2 x - \underset{(\sec^2 x - 1)}{\tan^2 x})\,dx$$

$$= \int \tan^2 x \sec^2 x - \int \sec^2 x^{dx} - \int dx.$$

$$= \frac{\tan^3 x}{3} - \tan x - x + C.$$

* products of Sines & Cosines

1) * $\int \sin mx \sin nx\,dx = \frac{1}{2}\int [\cos(m-n)x - \cos(m+n)x]\,dx$

2) $\int \sin mx \cos nx\,dx = \frac{1}{2}\int [\sin(m-n)x + \sin(m+n)x]\,dx$

3) $\int \cos mx \cos nx\,dx = \frac{1}{2}\int [\cos(m-n)x + \cos(m+n)x]\,dx$.

eg.

* $\int \sin 3x \cos 5x\,dx = \frac{1}{2}\int [\sin(-2x) + \sin(8x)]\,dx$

$$= \frac{1}{2}\int (\sin(8x) - \sin(2x))\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x + \frac{1}{2}\cos 2x\right] + C.$$