Integrals (Integrations)

* Indefinite integral

Def: A differentiable function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x) \quad \forall\, x \in I$$

The antiderivative function is called indefinite integral and is denoted by

$$\text{Integral sign} \longrightarrow \int \underbrace{f(x)}_{\text{Integrand}}\, d\underline{x} \longrightarrow \text{variable of integration}$$

e.g. Find the antiderivative of $f(x) = 2x$.

we now that $F(x) = x^2$,

But $F(x) = x^2 + 1$

and $F(x) = x^2 - 2$

$F(x) = x^2 + 107$

are also antiderivative of $f(x) = 2x$ because $F'(x)$ in all cases is $2x = f(x)$.

Therefore, we need to know the general form of antiderivative of f. The most general anti-derivative of f is

$$F(x) + c, \text{ where } c \in \mathbb{R}.$$

That is

$$\int f(x)\, dx = F(x) + c.$$

* properties of Indefinite Integrals:

1. $\int k\, f(x)\,dx = k \int f(x)\,dx \quad \left(\int k\,dx = kx + C\right)$
2. $\int [f(x) \mp g(x)]\,dx = \int f(x)\,dx \mp \int g(x)\,dx$

* Some examples …

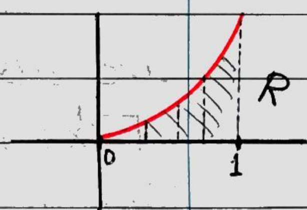
1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1, \; n \in \mathbb{Q}$
2. $\int \sin(x)\,dx = -\cos(x) + C \quad \left(\int \sin(kx)\,dx = \frac{-1}{k}\cos(kx) + C\right)$
3. $\int e^x\,dx = e^x + C$
4. $\int \frac{1}{x}\,dx = \ln(x) + C.$

For more examples, see the sheet I gave it to you.

* Definite Integrals:

Before defining this type of integrals, we need to give some knowledge about the area of some region.

Let's start from an example: What is the area of the parabola $f(x) = x^2$ from $\underline{0}$ to $\underline{1}$.



The question is: what is the area of region R

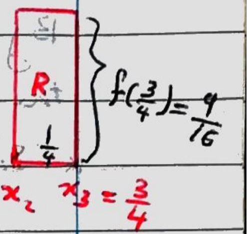
* step 1: Divide $[0,1]$ into $\underline{4}$ equal parts R_1, R_2, R_3 & R_4. Eeach R_i looks like a rectangle with no head.

* step 2: Draw a line around the head of each R_i as the same as its base, and compute the area of each R_i by

rule base $\times$ height. That is

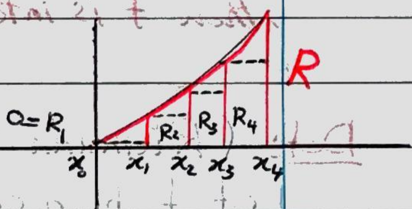
$area(R) \leq area(R_1) + area(R_2) + area(R_3)$
$+ area(R_4) \longrightarrow$ This is upper sum

Let $A(R)$ be the area of R (from upper)



$$\Rightarrow A(R) \leq A(R_1) + A(R_2) + A(R_3) + A(R_4)$$
$$= \frac{1}{4} \times \left(\frac{1}{4}\right)^2 + \frac{1}{4} \times \left(\frac{1}{2}\right)^2 + \frac{1}{4} \times \left(\frac{3}{4}\right)^2 + \frac{1}{4} \times (1)^2$$
$$= 0.46$$

To find the lowersum, draw the head to each R_i under the curve $f(x) = x^2$. That is



$$\Rightarrow A(R) \geq \frac{1}{4} \cdot 0^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2$$
$$= 0.21$$

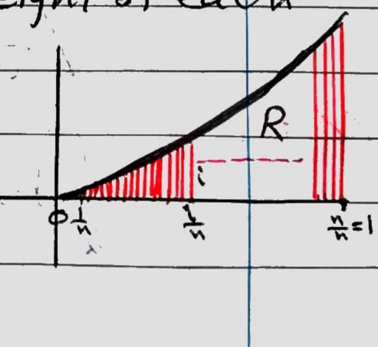
Therefore, $\underbrace{0.21}_{A_L} \leq A \leq \underbrace{0.46}_{A_U}$

* When dividing $[0,1]$ (our Region) into 1000 parts, we get

$$0.332 \leq A(R) \leq 0.3338$$

By repeating this procedure with a larger number of parts (strips), we get an accurate result.

* Let us divide $[\overset{a}{0}, \overset{b}{1}]$ into $\underline{n}$ equal strip R_i with width $\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$. Note that the Height of each R_i is $f(\frac{i}{n})$.



$$\Rightarrow A(R) \leq \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \cdots + \frac{1}{n} f\left(\frac{n}{n}\right)$$

$$= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$\Rightarrow A_U(R) \leqslant \frac{1}{n}\cdot\frac{1}{n^2}\left(1^2+2^2+3^2+\cdots+n^2\right)$$

$$= \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \frac{1}{6n^2}(n+1)(2n+1)$$

$$= \frac{2n^2+3n+1}{6n^2}$$

as $n \to \infty$, $A_U(R) = \frac{2}{6} = \frac{1}{3}$.

By the same way, we can obtain the same result from the Lower sum $A_L(R)$. $\left(A_L(R) = \frac{1}{3}\right)$

* Since $A_L(R) \leqslant A(R) \leqslant A_U(R) \Longrightarrow A(R) = \frac{1}{3}$

* *If the limit of upper and Lower sum are equal, as $n \to \infty$, then f is integrable.*

Def: (Riemann Definition of Integration)

Let f be a continuous function on $I=[a,b]$ and $P=\{x_0, x_1, \ldots, x_n\}$ be any partition of I. The definite Integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x .$$

This means that for each $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall P$ of $[a,b]$ with $\|P\| < \delta$ and $\forall x_i^* \in [x_{i-1}, x_i]$, we have

$$\left| \sum_{i=1}^{n} f(x_i^*)\,\Delta x - \int_a^b f(x)\,dx \right| < \epsilon .$$

Note!

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt \quad (\text{variable is not a problem})$$

Thm: Every continuous function is integrable.

note that not all function are integrable

eg: $f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \in \mathbb{Q}^c \end{cases}$

$U = 1 \ \& \ L = 0 \Rightarrow L \neq U$.

Example

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum f(x_i)\,\Delta x$$

$\Delta x = \frac{b-a}{n}$
$= \frac{3}{n}$

$x_1 = x_0 + \Delta x$
$0 + \Delta x = \frac{3}{n}$

$x_2 = 2\Delta x = \frac{3\times 2}{n}$

$\boxed{x_i = \frac{3i}{n}}$

$$= \lim \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18 i}{n}\right]$$

$\left(\frac{n(n+1)}{2}\right)^2$ $\qquad \frac{n(n+1)}{2}$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = \boxed{-\frac{27}{4}}$$

This integral cannot be interpreted as an area because f takes both +ve & −ve values. (we can say this is a difference of areas).

$f(x) \geq 0$

Def: If f is non-negative ~~and~~ integrable over $I=[a,b]$, then the area under the curve $\boxed{y=f(x)}$ over I is

$$A = \int_a^b f(x)\,dx.$$

* properties of Definite Integral:

Let f & g be integrable on $[a,b]$.

1. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$, $\boxed{\Delta x \to 0}$

3. $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

4. $\int_a^b [f(x) \mp g(x)]\,dx = \int_a^b f(x)\,dx \mp \int_a^b g(x)\,dx.$

5. if $c \in [a,b]$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

6. if f has max & min values on $[a,b]$, then

$$\min(f)\cdot(b-a) \leq \int_a^b f(x)\,dx \leq \max(f)\cdot(b-a).$$

7. if $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

The Fundamental Theorem of Calculus: F.T.C.

Thm (The F.T.C. part 1) FTC1

If f is cont on $[a,b]$ then $F(x) = \int_a^x f(t)\,dt$ is cont on $[a,b]$ & differentiable on (a,b) and its derivative $F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

e.g. use F.T.C. to find

$\frac{d}{dx}\int_a^x \cos t\,dt$.

Thm (The F.T.C. part 2) FTC2

If f is cont on $[a,b]$ & F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$