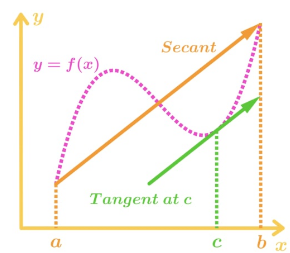
1 Some Results on Rolle's and Mean Value Theorems

Theorem 1.1 (**Rolle's Theorem**)**.** *Let a function f be continuous over $[a, b]$ and differentiable over (a, b) (why open interval) such that $f(a) = f(b)$, then there exists a point c where $a < c < b$ and $f'(c) = 0$.*

Theorem 1.2. *Let a function f be continuous over $[a, b]$ and differentiable over (a, b) (why open interval). Then there exists a point c where $a < c < b$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Result 1.3. *Let f be defined on $[a, b]$. If $f'(c) > 0$ for all $c \in [a, b]$, then f is increasing.*

Proof. Let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$. By MVT, $\exists \quad c \in (a, b)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \Rightarrow f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(c) > 0$, so

$$f(x_2) - f(x_1) > 0 \Rightarrow f(x_2) > f(x_1), \quad f \text{ is increasing.}$$

□

The converse of this result is not true. For example, take $f(x) = x^3$ on $[-1, 1]$. This f is increasing, but $f'(0) = 0$.

Definition 1.4. *A function f is called non-increasing on $[a, b]$ if $x_1 < x_2$, then $f(x_1) \le f(x_2)$ for all $x_1, x_2 \in [a, b]$. It is called non-decreasing on $[a, b]$ if $x_1 < x_2$, then $f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in [a, b]$.*

Result 1.5. *If f is non-increasing on $[a, b]$, then $f'(x) < 0 \quad \forall x \in [a, b]$.*

Proof. Let (x, y) be a point on the curve $y = f(x)$. Let $h > 0$, then $x + h > x$. Since f is non-increasing,

$$f(x + h) \le f(x) \Rightarrow \frac{f(x + h) - f(x)}{h} \le 0$$

Taking the limit,

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \le 0 \Rightarrow f'(x) \le 0$$

Thus, if f is non-increasing at x, then $f'(x) \le 0$.

□

Example 1.6. *Show that Rolle's Theorem (whether) is valid for $f(x) = x^3 - 3x^2 + 2x$ on $[0, 2]$.*

Solution. *To apply Rolle's Theorem:*

- *f is continuous on $[0, 2]$ and differentiable on $(0, 2)$.*
- *$f(0) = f(2) = 0$.*

Since $f(0) = f(2)$, we now apply the theorem:

$$f'(x) = 3x^2 - 6x + 2$$

Setting $f'(c) = 0$:

$$3c^2 - 6c + 2 = 0$$

Solving for c:

$$c = \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{6 \pm \sqrt{12}}{6} = \frac{6 \pm 2\sqrt{3}}{6} = 1 \pm \frac{\sqrt{3}}{3} \in (0, 2).$$

Example 1.7. *Use Rolle's Theorem to show that* $f(x) = x^7 + 5x^5 + x^3$ *has only one real root in* $[0, 1]$*.*

Solution. *By IVT, there exists at least one* $c \in (0, 1)$ *such that* $f(c) = 0$*. Suppose* f *has two roots* $x_1, x_2 \in [0, 1]$*. By Rolle's Theorem (as* $f(x_1) = f(x_2) = 0$*), there exists* $c \in (x_1, x_2)$ *such that:*

$$f'(x) = 7x^6 + 25x^4 + 3x^2 + 1 > 0$$

which contradicts the assumption of two roots. Thus, f *has at most one real root in* $[0, 1]$*.*

Example 1.8. *Prove that* $|\sin x - \sin y| < |x - y|$ *for all* $x \neq y$ *(Use MVT).*

Solution. *Define* $f(x) = \sin x$*.*

Start with the case $y > x$*:*

Note that $f(x)$ *is everywhere continuous and differentiable, in particular in the interval* $[x, y]$ $(y > x)$*. By MVT* $\exists$ *a point* $c \in (x, y)$ *such that*

$$f(y) - f(x) = f'(c) \cdot (y - x) \ ,$$

i.e.

$$\sin y - \sin x = \cos c \cdot (y - x) \qquad \Rightarrow \qquad |\sin y - \sin x| = |\cos c| \cdot |y - x| \ .$$

But $-1 < \cos c < 1 \iff |\cos c| \leq 1$*, hence*

$$|\sin y - \sin x| = |\cos c| \cdot |y - x| \leq |y - x|.$$

For the case $x > y$ *the argument is analogous to the* $y > x$ *case.*