Derivatives:

Def: The derivatives of a function f at a number "a", denoted by $f'(a)$, is

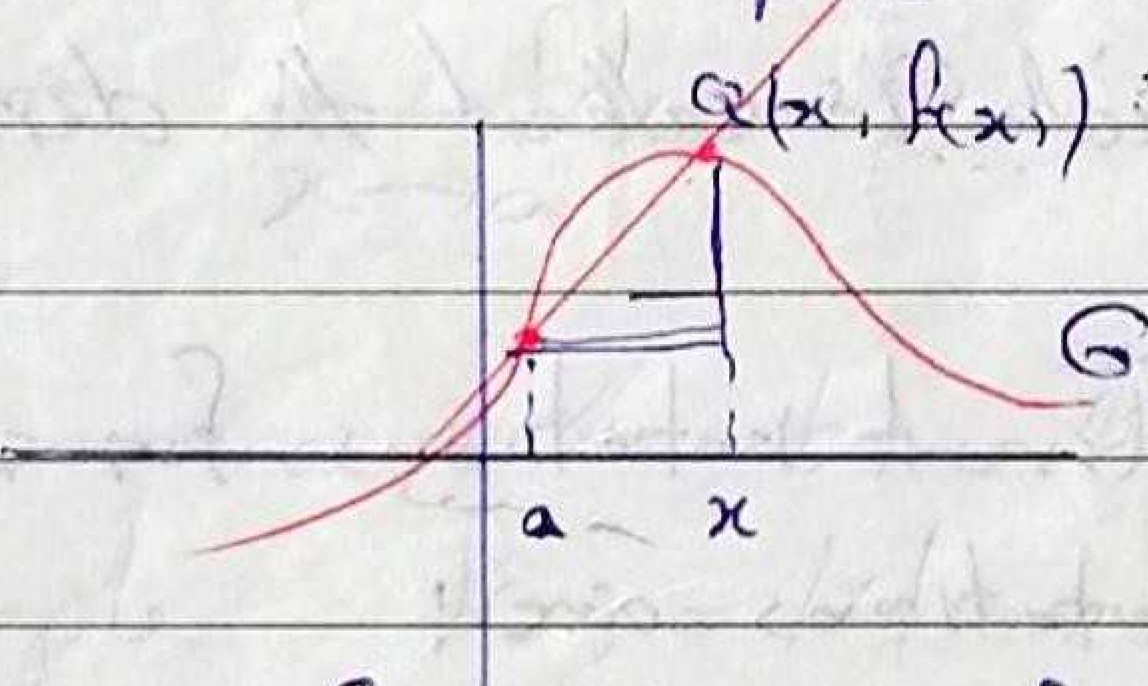
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{if the limit exists.}$$

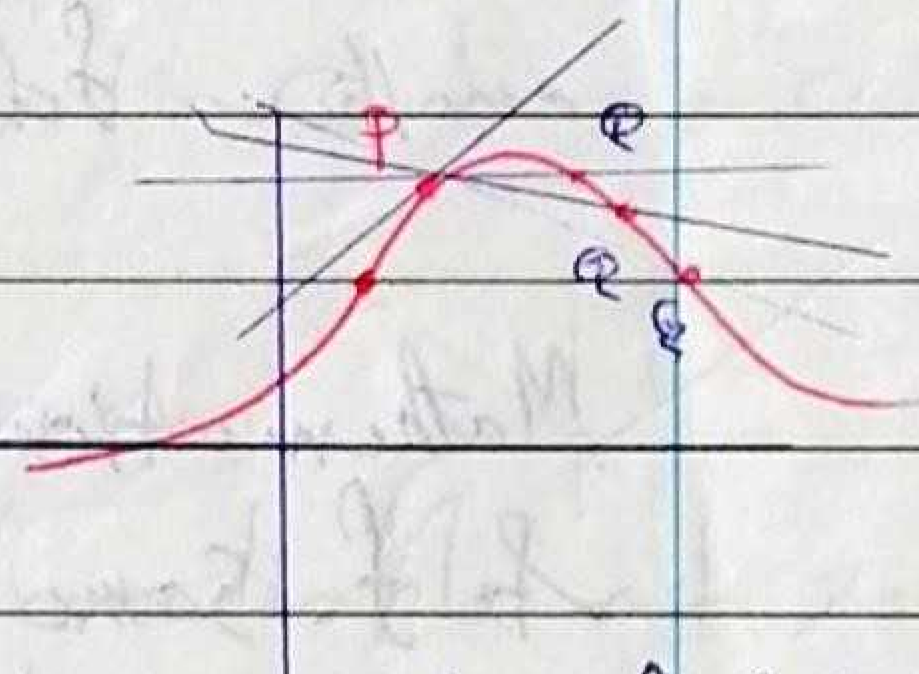
If we write $x = a+h \Rightarrow h = x-a$ and $h \to 0$ iff $x \to a$. Substituting these variables into the above formula we get.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

This looks like the slope of a line at $(a, f(a))$.

Geometric interpretation of the derivative





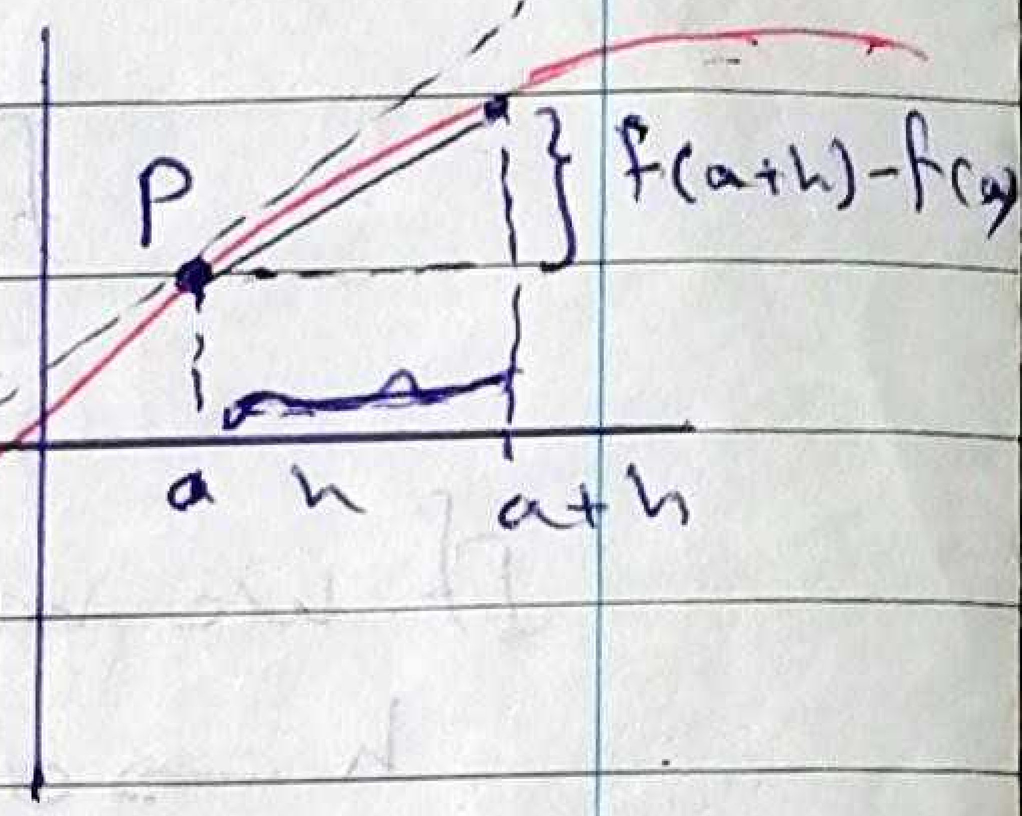
Let $y = f(x)$ be a function, we need to find the tangent line to the graph $\underset{=}{G}$ of f at the point $(a, f(a))$ then we consider a point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ

$$m_{PQ} = \frac{f(x) - f(a)}{x-a}$$

The tangent line to the curve $y = f(x)$ at the point

$p(a, f(a))$ is the line passes through p with the slope

$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$. This is exactly the derivative of f at "a".



D $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

= slope of the tangent at p.

In some books, you may find Δx as h.

* If we replace "a" by "x" (a variable), ~~the~~ we get

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. This gives us

another function which is called the derivative of f.

Mathematicians use different notations for the derivative of f. Some of these notations are:

$f'(x)$, y', $\frac{df}{dx}$, $\frac{dy}{dx}$, $\frac{d}{dx}(f(x))$, $D_x(y)$.

Def: A function f is differentiable at "a" if $f'(a)$ exists.
It is differentiable on an open interval I if it is differentiable at every point in I.

negation discont $\Longrightarrow$ not differentiability

Thm: If f is differentiable at "a", then f is continuous at "a"

proof: To prove that f is cont, we need to show that $\lim_{x\to a} f(x) = f(a)$. This ... will be done by showing that the difference $f(x)-f(a)$ approaches 0.

suppose that f is differentiable at "a". That is

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ exists.}$$

Now consider

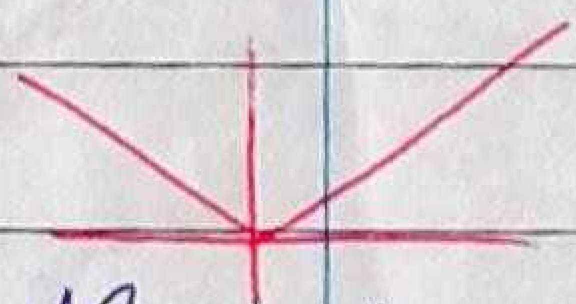
$$f(x)-f(a) = \frac{f(x)-f(a)}{x-a}(x-a). \quad \text{take limit}$$

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)\right]$$

$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a).$$

$$= f'(a)\cdot 0 = 0.$$

$\Longrightarrow \lim_{x\to a} f(x) = f(a)$. Thus f is cont at "a".



The converse of the above Thm is not true.

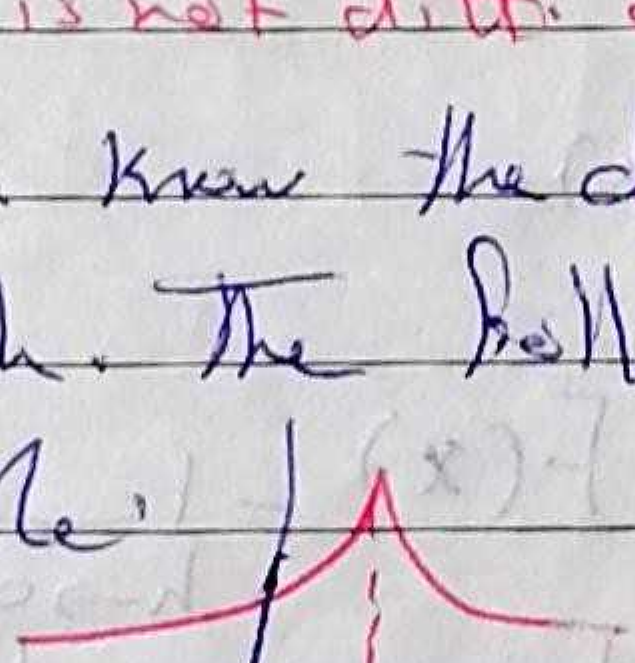
example: $f(x) = |x|$ is not differentiable at $x=0$. but cont. at $x=0$

one sided derivative

$f(x)=\sqrt{x}$, f is not diff. at "0". $f'(0) = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty$. vertical tangent at 0.

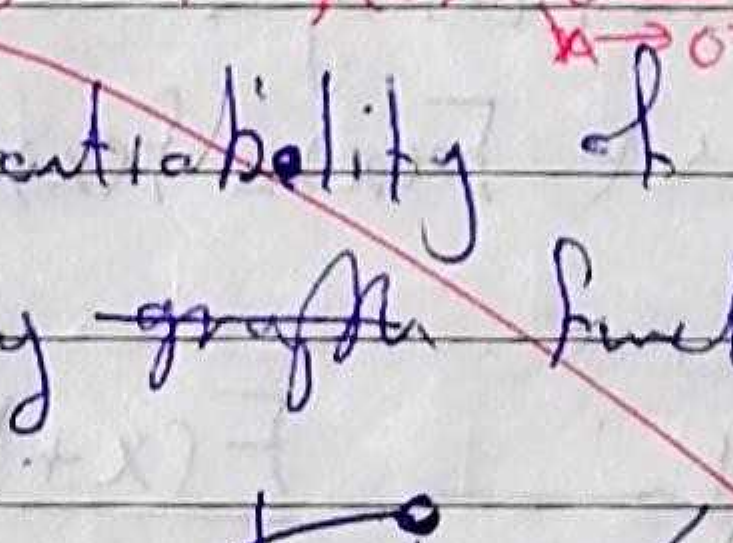
* One can know the differentiability of a function from its graph. The following ~~graph~~ function are not differentiable:

(a)



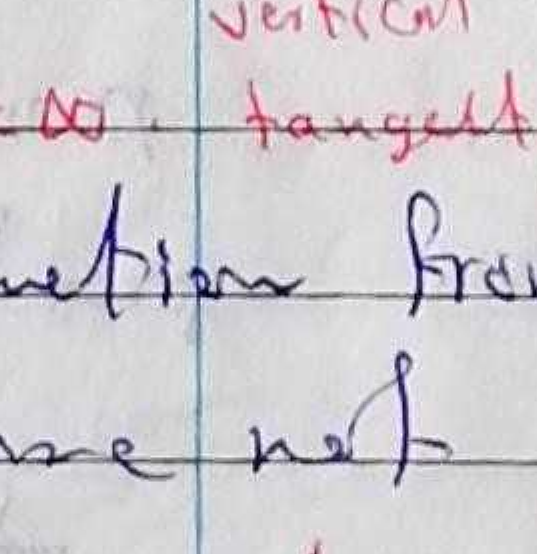
corner "Kink"

(b)



discontinuity

(c)



vertical tangent

example

Differentiation Rules:

Let f, g be two differentiable functions, and c any real number.

① $\frac{d}{dx}(\overset{f(x)=c}{c}) = 0$ and $\frac{d(\overset{f(x)=x}{x})}{dx} = 1$

② $\frac{d}{dx}(x^n) = n x^{n-1}$, $n > 0$.

③ $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{df}{dx}$

④ $\frac{d}{dx}(f(x) \mp g(x)) = \frac{df}{dx} \mp \frac{dg}{dx} = \frac{d}{dx}(f(x)) \mp \frac{d(g(x))}{dx}$

⑤ $\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}(g(x)) + g(x) \cdot \frac{d}{dx}(f(x))$

$= f \cdot g' + g \cdot f'$.

$\boxed{\mp f(x) \cdot g(x)}$ ⑥ $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \frac{d}{dx}(f(x)) - f(x) \cdot \frac{d}{dx}(g(x))}{[g(x)]^2}$.

$= \frac{g \cdot f' - f \cdot g'}{g^2}$.

we only give proof to ⑤, the others can be proved similarly.

proof(⑤) Let $F(x) = f(x) \cdot g(x)$.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) g(x)}{h}$$

~~In order to compute this limit, we need to separate f & g~~ as in the proof of the Sum Rule. We add & subtract the term $f(x+h) g(x)$ in the numerator.

$$\rightarrow F'(x) = \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h} + g(x)\,\frac{f(x+h)-f(x)}{h}\right]$$

$$= \lim_{h\to 0} f(x+h)\cdot \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\cdot \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= f(x)\cdot g'(x) + g(x)\cdot f'(x) = f\cdot g' + g\cdot f'.$$

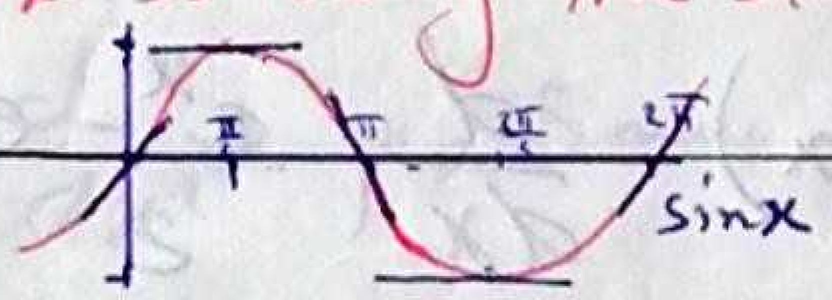
Some Basic Derivatives:

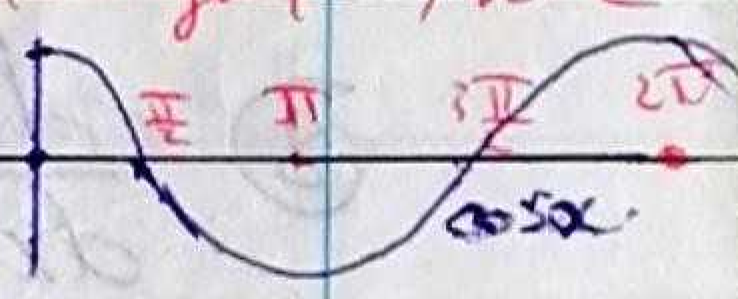
① $f(x) = \sqrt{x} \implies f'(x) = \frac{1}{2\sqrt{x}}$

② $f(x) = \sin(x) \implies f'(x) = \cos(x)$. By def.

$$f'(x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \cos x.$$

By the graph of $\sin x$ & using the slope of tangent, we can get $\cos x$.





③ $f(x) = \cos(x) \implies f'(x) = -\sin x$

④ $f(x) = \tan x \implies f'(x) = \sec^2 x$

⑤ $f(x) = e^x \implies f'(x) = e^x$.

$\frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\csc x) = -\csc x \cot x$.

By Def.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} = e^x \underbrace{\lim_{h\to 0}\frac{e^h - 1}{h}}_{1}$$

Chain Rule: If $y = f(u)$ & $u = g(x)$ are differentiable functions, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}}$$

⑥ $f(x) = \ln x \Longrightarrow f'(x) = \frac{1}{x}$.

We know that $\frac{d(x)}{dx} = 1$, $x = e^{\ln x}$,

$$\frac{d}{dx}(e^{\ln x}) = e^{\ln x}\frac{d}{dx}(\ln x) = 1$$

$$\Rightarrow x \cdot \frac{d}{dx}(\ln x) = 1 \Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

* $\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$.

⑦ $\frac{d}{dx}(a^x) = a^x \ln a$. exp: $(a^x)' = \lim_{h\to 0}\frac{a^{x+h} - a^x}{h}$
$= a^x \lim_{h\to 0}\frac{a^h - 1}{h}$ $= \ln a$.

or: $\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{\ln a})^x$

$$\boxed{\frac{d}{dx}(a^{g(x)}) = a^{g(x)}\ln a \cdot g'(x)} = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot[x \cdot 0 + \ln a \cdot 1] = a^x \ln a$$

$e^{x\ln a}\cdot\frac{d}{dx}(x\ln a)$

⑧ $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$.

$\frac{d}{dx}(\cosh x) = \sinh x$, $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$.

$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$, $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$.

Give some examples:

* ① $f(x) = \sqrt{\ln x}$, ② $f(x) = \tan(\cos x)$, $f(x) = \log_5\left(\frac{\sin x}{1-x}\right)$

* Implicit Differentiation:-

This method is a special case of the chain rule. Using this, we need to differentiable both sides of the equation with respect to x and then solving the resulting equation for y'.

e.g. $x^2 + y^2 = 16 \Rightarrow \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(16)$

$2x + 2yy' \Rightarrow y' = \frac{-x}{y}$.

H.W. find y' for ① $x^3+y^3 = 4xy$.

② $\cos(x+y) = y^2 \sin x$

③ $y = \sin^{-1} x$, $y' = \frac{1}{\sqrt{1-x^2}}$

③ $\sin y = x$

$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$

$\cos y . y' = 1 \Rightarrow y' = \frac{1}{\sqrt{\cos^2 y}}$

$\cos^2 y + \sin^2 y = 1 \Rightarrow \cos^2 y = 1 - x^2$

* Higher Derivatives:

Given $y = f(x)$, the second derivative of f is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2}$$, Some other notations

$f''(x)$, or $D^2 f(x)$. and $f^{(n)}(x)$ for higher than 3.

e.g. find $f''(x)$ of $f(x) = x\cos x$.

e.g. $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$ for $f(x) = \frac{1}{x}$.

H.W find $D^{99} \sin(x)$?